Complexity of Polyadic Boolean Modal Logics: Model Checking and Satisfiability

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Background

Boolean modal logics

Polyadic Boolean modal logics

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Open problems

Binary vs. Polyadic

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Is there an inherent issue in using relations of higher arity? For example, is the complexity of basic reasoning problems that much harder with higher arity relations?

Deterministic sample of recent results

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- (Bednarczyk, 2021) considered *guarded forward fragment* GFF, which is a polyadic extension of $\mathcal{ALC}$ with global diamond. Complexities of satisfiability and CQ-entailment coincide.

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This paper: New results on the complexity of polyadic Boolean modal logics!

Boolean modal logics

Main idea: Diamonds can contain Boolean combinations of relations.

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2. $\mathfrak{M}, w \Vdash \langle R \cap S \rangle p$ iff there exists $(w, v) \in (R^{\mathfrak{M}} \setminus S^{\mathfrak{M}})$ s.t. $\mathfrak{M}, v \Vdash p$.

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Theorem (Lutz & Sattler, 2000)

- *The satisfiability problem of* $\mathrm{ML}(\neg)$ *is* ExpTime*-complete.*
- *The satisfiability problem of* $\mathrm{ML}(\neg, \cap)$ *is* NExpTime*-complete.*

Further operators

Lets throw in two additional operators: s and I.

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Theorem (Lutz et. al., 2001)

1. $\mathrm{ML}(I, s, \neg, \cap)$ *is equi-expressive with* FO^2 *on the level of sentences.*
2. *Limit attention to formulas in which at most c binary relations occur, c being a fixed constant. Then the satisfiability problem of* $\mathrm{ML}(I, s, \neg, \cap)$ *is* EXPTIME*-complete.*

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Fact

The satisfiability problem for $\mathrm{PML}(\sigma, \neg, \cap)$ *is* NEXPTIME*-complete.*

New complexity results

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3. *Let* $c \geq 0$ *be a fixed constant. Limit attention to those formulas of* $\mathrm{PML}(\sigma, \neg, \cap)$ *in which at most* c *relations occur and the arity of each relation is at most* c*.*

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Main technique: reduce the satisfiability problems of $\mathrm{PML}(\neg)$ and $\mathrm{PML}(\sigma, \neg, \cap)$ to that of $\mathrm{PML} + \langle E \rangle$.

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$$\eta := \bigwedge_{\substack{\langle R \rangle(\psi_1,\ldots,\psi_k), \\ \langle \overline{R} \rangle(\chi_1,\ldots,\chi_k) \\ \in \mathrm{Subf}(t(\varphi))}} \Big(\langle E \rangle (\neg \langle R \rangle(\psi_1, \ldots, \psi_k) \wedge \neg \langle \overline{R} \rangle(\chi_1, \ldots, \chi_k)) \rightarrow \bigvee_{1 \leq \ell \leq k} \neg \langle E \rangle (\psi_\ell \wedge \chi_\ell) \Big).$$

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Thanks! :–)