Expressing boundedness in static computational logic

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Overview

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Syntax of SCL

Definition

Fix a countable set $\mathrm{LBS} = \{L_n \mid n \in \mathbb{N}\}$ of label symbols. For reach (relational) vocabulary τ the set of formulas $\mathrm{SCL}[\tau]$ is defined by

$$\phi ::= x = y \mid R(\overline{x}) \mid C_L \mid \neg\phi \mid \phi \wedge \phi \mid \exists x \phi \mid L\phi,$$

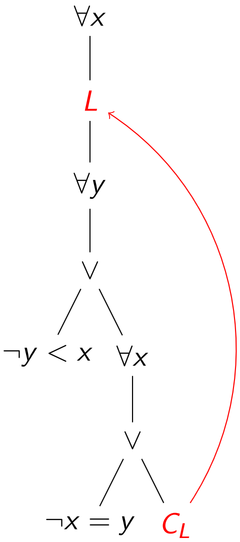
where $R \in \tau$ and $L \in LBS$.

Evaluation games

- The semantics of SCL are defined in terms of evaluation (or semantical) games $\mathcal{G}_\infty(\mathfrak{A}, s, \varphi)$.
- These games have two players, the Verifier and Falsifier.
- If the Verifier has a winning strategy in $\mathcal{G}_\infty(\mathfrak{A}, s, \varphi)$, then we write

$$\mathfrak{A}, s \models_\infty \varphi$$

How to define well-foundedness



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LFP_1

Definition

The logic LFP_1 is defined via the following formula construction rules.

- Every FO-formula is in LFP_1.
- If X is a k-ary relation symbol, $\varphi(x_1, \ldots, x_k)$ is an FO-formula in which X occurs only positively and $u_1, \ldots, u_k$ are variables, then $[\mathrm{LFP}_{X,\overline{x}}\varphi]\overline{u}$ is in LFP_1.
- LFP_1 is closed under $\wedge, \vee, \exists, \forall$.

Theorem (Kleene 1955, Spector 1961)

LFP_1*-definable relations over* $(\mathbb{N}, +, \cdot, 0, 1)$ *coincide with the* Π^1_1*-relations.*

Theorem (Immermann 1982)

$\mathrm{LFP}_1 \equiv \mathrm{LFP}$ *over finite structures.*

Expressive power of SCL

- Formulas of LFP_1 can be translated to SCL in a straightforward way. E.g.

$$[\mathrm{LFP}_{X,x,y}(E(x,y) \vee \exists z(E(z,y) \wedge X(x,z)))]xy$$

translates to

$$L(E(x,y) \vee \exists z(E(z,y) \wedge \exists y(y = z \wedge C_L)))$$

Theorem

$\mathrm{SCL} \equiv \mathrm{LFP}_1$

- It follows that $\mathrm{SCL} < \forall\mathrm{SO}$.
- A result of Kozen implies that over countable structures we have that

$$\mathrm{SCL} + \text{“dictionaries”} \equiv \forall\mathrm{SO}$$

IND programming language (Harel & Kozen, 1984)

Algorithm 1 IND-program corresponding to the sentence

$$\forall x L \forall y (y < x \to \forall x (x = y \to C_L)).$$

```
1: Universally choose x
2: Universally choose y
3: if y ≥ x then
4:     accept
5: end if
6: Universally choose x
7: if x ≠ y then
8:     accept
9: end if
10: goto Step 2
```

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Reachability games

Definition

A reachability game is a two-player game played on a directed graph $G = (V, E)$. The game is defined as follows:

- The vertices V are partitioned into V_1 and V_2, where V_1 consists of the vertices controlled by Player 1 (in this talk Verifier) and V_2 consists of the vertices controlled by Player 2 (Falsifier).
- The game starts at an initial vertex $v_0 \in V$, and the players take turns choosing edges to traverse, based on the partitioning of V.

A winner of a finite (maximal) play is determined as follows: if the final node is controlled by the Verifier, Falsifier wins, and otherwise the Verifier wins. Neither player wins infinite plays.

Rank of a game

- Let $\mathcal{G}$ be a reachability game and let σ be a winning strategy for $P \in \{\text{Verifier}, \text{Falsifier}\}$. The set of plays of $\mathcal{G}$ where P follows σ is a well-founded tree.
- We define $\mathrm{rank}(\sigma)$ as the height of this tree (an ordinal).
- Furthermore, we define

$$\mathrm{rank}_P(\mathcal{G}) := \min\{\mathrm{rank}(\sigma) \mid \sigma \text{ is a winning strategy for } P\}.$$

If P does not have a winning strategy in $\mathcal{G}$, then $\mathrm{rank}_P(\mathcal{G}) = \infty$.

Bounded SCL (BndSCL)

- Let $\mathfrak{A}$ be a structure, s an assignment φ a sentence of SCL. Given an ordinal α we write
$$\mathfrak{A}, s \models_{\alpha} \varphi$$
iff the Verifier-rank of $\mathcal{G}_{\infty}(\mathfrak{A}, s, \varphi)$ is $< \alpha$.
- BndSCL is a semantical variant of SCL obtained by replacing
$$\models_{\infty} \text{ with } \models_{\omega}$$

Example

Example

Consider the sentence

$$\varphi := \forall x \forall y (x = y \vee L(E(x,y) \vee \exists z (E(x,z) \wedge \exists x (x = z \wedge C_L))))$$

of SCL and a graph G.

- $G \models_\infty \varphi$ iff G is connected.
- $G \models_\omega \varphi$ iff there exist $n \in \mathbb{N}$ such that the distance between any two nodes is at most n.

Main result

Theorem

BndSCL $<$ SCL.

Earlier results on bounded recursion

- Goranko, Hella, Kuusisto and Rönnholm have studied extensively bounded variants of standard modal logics with recursion, such as $\mathrm{CTL}, \mathrm{ATL}$ and μ-calculus.
- A common theme has been that the bounded variants are incomparable with the original logics with respect to expressive power.
- Furthermore, while the original logics have the finite model property, the bounded variants no longer enjoy it.

Earlier results on bounded recursion

- Given a pointed Kripke model $(\mathfrak{M}, w)$, we say that it has the **bounded dead-end property**, if there exists $n \in \mathbb{N}$ such that no matter where we go from w we will eventually reach a dead-end in at most n-steps.

Proposition (Folklore)

The class of pointed Kripke-models which have the bounded dead-end property is not definable in MSO.

- This class is definable if we can universally quantify over binary relations.

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Comparing ranks of reachability games

- Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be reachability games. We can define a new game $\mathcal{I}(\mathcal{G}_1, \mathcal{G}_2)$ as follows.
  - Two players, Verifier and Falsifier.
  - Positions are pairs (v, v'), where v is a position in $\mathcal{G}_1$ and v' is a position in $\mathcal{G}_2$.
  - Given a position (v, v'), the players first make a move in $\mathcal{G}_1$ and then in $\mathcal{G}_2$ giving us the next position (u, u').
  - Game ends when one of the nodes in (v, v') is a dead-end.

Proposition

Suppose that

$$\mathrm{rank}_{\mathit{Falsifier}}(\mathcal{G}_1) \leq \mathrm{rank}_{\mathit{Verifier}}(\mathcal{G}_2) \text{ and } \mathrm{rank}_{\mathit{Falsifier}}(\mathcal{G}_1) < \infty.$$

Then Falsifier has a winning strategy in $\mathcal{I}(\mathcal{G}_1, \mathcal{G}_2)$.

Comparing ranks of reachability games

- Let us modify the winning conditions of the Verifier in $\mathcal{I}(\mathcal{G}_1, \mathcal{G}_2)$: if the players reach a position (v, v'), where one of the nodes is a dead-end, the winner is determined as follows.
  - If either v or v' is a dead-end where it is Falsifier's turn to make a move, Verifier wins.
  - Otherwise the Falsifier wins.

Proposition

Suppose that

$$\mathrm{rank}_{\mathit{Falsifier}}(\mathcal{G}_1) = \mathrm{rank}_{\mathit{Verifier}}(\mathcal{G}_2) < \omega$$

Then Verifier has a winning strategy in $\mathcal{I}(\mathcal{G}_1, \mathcal{G}_2)$.

Comparing ranks of reachability games

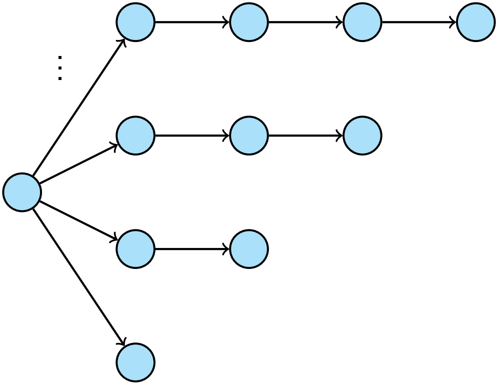
Proposition

Suppose that

$$\omega \leq \mathrm{rank}_{\mathit{Falsifier}}(\mathcal{G}_1) = \mathrm{rank}_{\mathit{Verifier}}(\mathcal{G}_2) < \infty$$

Then Falsifier has a winning strategy in $\mathcal{I}(\mathcal{G}_1, \mathcal{G}_2)$.

Comparing ranks of reachability games



$\mathrm{BndSCL} \leq \mathrm{SCL}$

- Given two sentences φ and ψ of SCL, it is straightforward to write a sentence $[\varphi : \psi]$ of SCL with the property

$$\mathfrak{A} \models_\infty [\varphi : \psi] \Leftrightarrow$$

$$\text{Verifier has a winning strategy in } \mathcal{I}(\mathcal{G}_\infty(\mathfrak{A}, \varphi), \mathcal{G}_\infty(\mathfrak{A}, \psi)).$$

- Given a sentence φ of SCL, we can construct another sentence φ^d such that the Falsifier-rank of $\mathcal{G}_\infty(\mathfrak{A}, \varphi^d)$ is the same as the Verifier-rank of $\mathcal{G}_\infty(\mathfrak{A}, \varphi)$.
- Now

$$\mathfrak{A} \models_\omega \varphi \Leftrightarrow \mathfrak{A} \models_\infty \varphi \wedge [\varphi^d : \varphi].$$

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Recursively saturated structures

- An **elementary type** is a set Γ of FO-formulas which all have the same (finite) set of free variables.

Definition

A structure $\mathfrak{A}$ is called **recursively saturated**, if for every finite assignment r over A we have that each recursive elementary type Γ which is finitely consistent with $(\mathfrak{A}, r)$ is realized in $(\mathfrak{A}, r)$.

- Examples include $(\mathbb{Q}, <), (\mathbb{C}, +, \cdot)$, which are also ω-**saturated**.

Recursively saturated structures

Theorem

If $\mathfrak{A}$ *is recursively saturated, then for every assignment s and a formula* φ *of* SCL *we have that*

$$\mathfrak{A}, s \models_{\infty} \varphi \Leftrightarrow \mathfrak{A}, s \models_{\omega} \varphi$$

In particular, $\mathrm{BndSCL} \equiv \mathrm{SCL}$ *over recursively saturated structures.*

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α-bounded SCL

- Given an ordinal α, $\alpha\mathrm{SCL}$ is a semantical variant of SCL obtained by replacing
$$\models_{\infty} \text{ with } \models_{\alpha} .$$
- **Example:** let φ be a formula of SCL which defines the class of well-founded linear orders. Now $\mathfrak{A} \models_{\alpha} \varphi$ iff the order type of every descending sequence is less than α.
- **Question:** what is the smallest ordinal α for which $\alpha\mathrm{SCL} \not\leq \mathrm{SCL}$?

α-bounded computational logic (CL)

- CL is obtained from SCL by extending with the capability to "modify" the underlying structure.
- For each $\alpha < \omega_1^{\mathsf{CK}}$ we have that $\alpha\mathrm{CL} \leq \mathrm{CL}$.
- **Question:** what is the smallest ordinal α for which $\alpha\mathrm{CL} \not\leq \mathrm{CL}$?

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Lindström's second theorem

Theorem (Lindström, 1969)

Let $\mathcal{L}$ be an effective regular logic such that $\mathrm{FO} \leq_{\mathit{eff}} \mathcal{L}$. Suppose that $\mathcal{L}$ has the countable downwards Löwenheim-Skolem property and its validity problem is recursively enumerable. Then $\mathcal{L} \equiv_{\mathit{eff}} \mathrm{FO}$.

Projective negation

- SCL is not a regular logic, because it is not closed under negation.

Definition

Let $\mathcal{L} = (L, \models_{\mathcal{L}})$ be an effective logic. We say that $\mathcal{L}$ has projective negation, if for every finite set $\tau \subseteq \mathrm{dom}(L)$ there exists a symbol set $\tau \subseteq \xi \subseteq \mathrm{dom}(L)$ and $\chi \in L(\xi)$ such that for every ξ-structure $\mathfrak{A}$ we have that

$$\mathfrak{A} \models_{\mathcal{L}} \chi \text{ iff } \mathfrak{A} \not\models_{\mathcal{L}} \varphi.$$

- SCL has projective negation, because the complement of an SCL-definable class is definable in $\exists\mathrm{SO}$.

An extension of Lindström's second theorem

Theorem

Let $\mathcal{L}$ be an effective semi-regular logic such that $\mathrm{FO} \leq_{\mathsf{eff}} \mathcal{L}$. Suppose that $\mathcal{L}$ has projective negation, the countable downwards Löwenheim-Skolem property and that its FO-consequences are axiomatizable. Then $\mathcal{L} \equiv \mathrm{FO}$.

- In particular, for every effective semi-regular logic $\mathcal{L}$ for which

$$\mathrm{FO} \leq_{\mathsf{eff}} \mathcal{L} \leq \mathrm{SCL},$$

we have that either $\mathrm{FO} \equiv \mathcal{L}$ or there is a sentence of $\mathcal{L}$ whose FO-consequences are not RE.

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Conclusions

- $\mathrm{BndSCL} < \mathrm{SCL} \equiv \mathrm{LFP}_1 < \forall\mathrm{SO}$.
- Over recursively saturated structures $\mathrm{BndSCL} \equiv \mathrm{SCL}$.
- There is no logic between FO and SCL which is stronger than FO and whose FO-consequences are axiomatizable.
- **Question:** what is the smallest ordinal α for which $\alpha\mathrm{SCL} \not\leq \mathrm{SCL}$?

Thanks!