Complexity classifications via algebraic logic

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Fragments of first-order logic

Classifying fragments $\mathcal{F}$ of first-order logic FO based on whether their satisfiability problem is decidable:

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Several fragments of FO known that have a decidable satisfiability problem:

monadic first-order logic, two-variable logic, guarded fragment, triguarded fragment, unary negation fragment, guarded negation fragment, uniform one-dimensional fragment, fluted logic, ordered logic, Maslov fragment, Herbrand fragment, positive first-order logic, ...

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In this work we present a novel approach towards classifying fragments of FO. Our approach is based on *relational algebras*.

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Given a set A, let $\mathrm{AD}(A)$ denote the set of all AD-relations over A. A k-**ary relational operator** F is a mapping (proper class) which associates to every set A a function F_A

$$F_A : \mathrm{AD}(A)^k \to \mathrm{AD}(A)$$

Examples

Equality e, which is 0-ary:
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Projection $\exists$:
$\exists_A(R) = \{(a_1, ..., a_{k-1})\} \mid (a_1, ..., a_k) \in R\}$

General relational algebras

Definition

Let $\mathcal{F}$ be a set of relational operators and let σ be a relational vocabulary. The set of terms $\mathrm{GRA}(\mathcal{F})[\sigma]$ is defined by the following grammar.

$$\mathcal{T} ::= R \mid F(\underbrace{\mathcal{T}, ..., \mathcal{T}}_{ar(F)\text{-times}}),$$

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Given a model $\mathfrak{A}$ of vocabulary σ and term $\mathcal{T} \in \mathrm{GRA}(\mathcal{F})[\sigma]$, its interpretation $[\![\mathcal{T}]\!]_{\mathfrak{A}}$ is defined recursively as follows.

1. $[\![R]\!]_{\mathfrak{A}} := (R^{\mathfrak{A}}, ar(R))$
2. $[\![F(\mathcal{T}_1, ..., \mathcal{T}_n)]\!]_{\mathfrak{A}} := F_A([\![\mathcal{T}_1]\!]_{\mathfrak{A}}, ..., [\![\mathcal{T}_n]\!]_{\mathfrak{A}})$

Algebraic characterisation of first-order logic

Each formula $\varphi(x_1, \dots, x_k)$ of FO (or any logic for that matter) defines in a natural way an AD-relation over each structure $\mathfrak{A}$:

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Theorem

$\mathrm{GRA}(e, p, s, I, \neg, J, \exists)$ *is equi-expressive with* FO.

Algebraic characterisations of fragments

The **suffix intersection** $\dot{\cap}$ is defined as follows: include in $(R \dot{\cap} S)$ every tuple of the higher-arity relation (say, R) that has as suffix a tuple of the smaller-arity relation.

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In all of these cases the translations are poly-time computable, so the complexities coincide.

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$\mathrm{GRA}(e, s, I, \neg, J, \exists)$ is decidable, *but* no tight upper bound on the complexity.

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$\mathrm{GRA}(e, s, \setminus, \dot{\cap}, \exists)$ *is* EXPTIME*-complete.*

Several results of this flavour were also established in (Jaakkola, 2021), where it was proven that e.g. $\mathrm{GRA}(s, \neg, \dot{\cap}, \exists)$ is Π^0_1-complete.

Some open problems

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Thanks! :–)