First-order logic with game-theoretic recursion

Reijo Jaakkola
reijo.jaakkola@tuni.fi

Tampere University

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- CL extends standard first-order logic FO with two natural features.

  1. The ability to modify the underlying model: adding new elements to the domain of the model, new tuples to relations, new relations etc.
  2. The ability to use recursion (looping) via self-reference.

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  Formulas are interpretated using *game-theoretical* semantics. We call the extension of FO with this type of recursion SCL (static computational logic).

Overview of the rest of the talk

1. Syntax & Semantics of SCL.
2. Validity problem of SCL.
3. Some results on the model theory of SCL.

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Fix a countable set $\mathrm{LBS} = \{L_n \mid n \in \mathbb{N}\}$ of *label* symbols. For reach relational vocabulary τ the set of formulas $\mathrm{SCL}[\tau]$ is defined by the following grammar:

$$\phi ::= x = y \mid R(\overline{x}) \mid C_L \mid \neg\phi \mid \phi \wedge \phi \mid \exists x \phi \mid L\phi,$$

where $R \in \tau$ and $L \in LBS$.

Game-theoretical semantics (GTS) for SCL

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  We call $\mathcal{G}_\infty$ the **unbounded evaluation game**.

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  - If the position is $(r, \alpha, +)$, where α is an atomic formula, then Verifier wins if $\mathfrak{A}, r \models \alpha$ and otherwise Falsifier wins.

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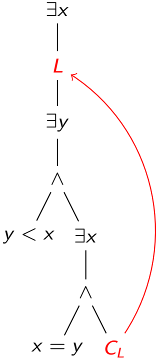

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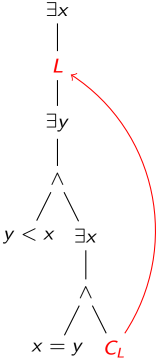

- Neither player wins infinitely long plays.

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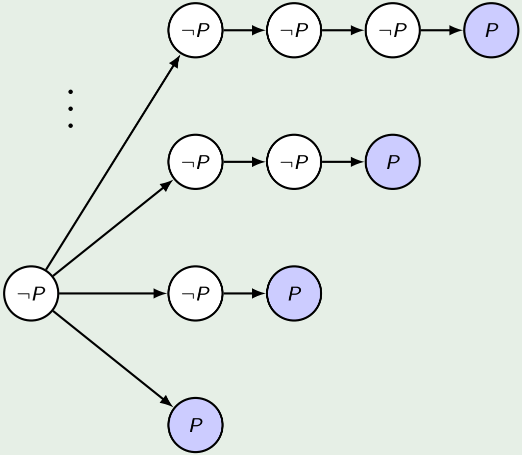
- The n-bounded evaluation game $\mathcal{G}_n$ works like $\mathcal{G}_\infty$ with the exception that looping atoms can be visited at most n times. If the players reach a looping atom after they have visited looping atoms n times, the game stops and neither player wins the game.

BndSCL vs SCL

Example



Consider the formula $\phi(x) := L(P(x) \vee \forall y(R(x, y) \rightarrow \exists x(x = y \wedge C_L)))$. Under unbounded semantics, ϕ is true at the "root" of the above structure, while under bounded semantics it is not true.

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Let ϕ be a formula of SCL. We define the ***n*th approximant** (or ***n*-approximant**) Φ^n_ϕ of ϕ to be the FO-formula obtained from the nth unfolding Ψ^n_ϕ by removing all the label symbols and replacing each occurrence of each looping atom by

1. $\bot$ if the occurrence of the atom is positive in Ψ^n_ϕ,
2. $\top$ if the occurrence is negative in Ψ^n_ϕ.

Example

Recall the formula $\phi := \exists x L \exists y(R(x,y) \wedge C_L)$. Since

$$\Psi^2_\phi := \exists x L \exists y(R(x,y) \wedge L \exists y(R(x,y) \wedge L \exists y(R(x,y) \wedge C_L)))$$

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we have that

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BndSCL as a fragment of $\mathcal{L}^{\omega}_{\omega_1\omega}$

Lemma

Let ϕ be a formula of SCL.

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Let ϕ be a formula of SCL. Then for every structure $\mathfrak{A}$ and assignment s we have that

$$\text{Verifier has a winning strategy in } \mathcal{G}_n(\mathfrak{A}, s, \phi) \Leftrightarrow \mathfrak{A}, s \models \Phi^n_\phi$$

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In particular, BndSCL $\leq \mathcal{L}^{\omega}_{\omega_1\omega}$.

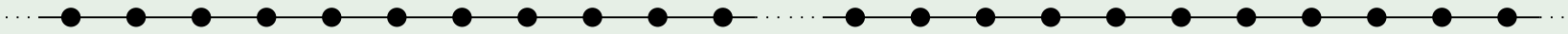
SCL $\not\leq$ BndSCL

Example

Connectivity of a graph is not definable in BndSCL. For example, consider the following graphs

$\mathfrak{G}_1$:

$\mathfrak{G}_2$:



These graphs are elementary equivalent, which implies that for every BndSCL sentence ϕ we have that if $\mathfrak{G}_1 \models \phi$, then $\mathfrak{G}_2 \models \phi$.

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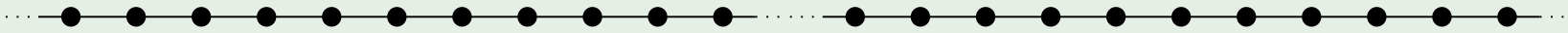

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Problem

Is BndSCL contained in SCL?

Valid sentences of BndSCL are RE

Theorem

Let ϕ be a sentence of BndSCL. Now ϕ is valid if and only if Φ_ϕ^n is valid, for some $n \in \mathbb{N}$.

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Proof.

($\Leftarrow$) If Verifier has a winning strategy in $\mathcal{G}_n(\mathfrak{A}, \phi)$, then they have a winning strategy in $\mathcal{G}_\omega(\mathfrak{A}, \phi)$.

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($\Rightarrow$) Suppose that $\neg\Phi^n_\phi$ is satisfiable, for every $n \in \mathbb{N}$. Since $\neg\Phi^n_\phi \models \neg\Phi^{n'}_\phi$, for every $n' < n$, using compactness we get that $\{\neg\Phi^n_\phi \mid n \in \mathbb{N}\}$ is satisfiable. Since ϕ is truth equivalent with $\bigvee_{n\in\mathbb{N}} \Phi^n_\phi$, there must exists a model $\mathfrak{A}$ such that Verifier does not have a winning strategy in $\mathcal{G}_\omega(\mathfrak{A}, \phi)$. $\square$

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($\Leftarrow$) If Verifier has a winning strategy in $\mathcal{G}_n(\mathfrak{A}, \phi)$, then they have a winning strategy in $\mathcal{G}_\infty(\mathfrak{A}, \phi)$.

($\Rightarrow$) If there exists for every $n \in \mathbb{N}$ a structure $\mathfrak{A}_n$ such that Verifier does not have a winning strategy in $\mathcal{G}_n(\mathfrak{A}_n, \phi)$, then one can use compactness theorem for FO to construct a structure $\mathfrak{A}$ such that Verifier does not have a winning strategy in $\mathcal{G}_\infty(\mathfrak{A}, \phi)$. $\square$

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Corollary

Valid sentences of BndSCL and SCL coincide.

Weakly complete axiomatization for SCL

- In our work we also developed a natural deduction style proof system which is weakly complete: if Σ is a set of FO-sentences and ϕ is a sentence of SCL, then $\Sigma \models \phi$ iff $\Sigma \vdash \phi$ in our system.

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$$\frac{\varphi[L\psi]}{\varphi[L\psi\{L\psi/C_L\}]} \updownarrow \text{(}L\text{Subst1)} \qquad \frac{\varphi[L\psi]}{\varphi[L\psi\{\psi/C_L\}]} \updownarrow \text{(}L\text{Subst2)}$$

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- The main idea is to show that for each formula ϕ we have that $\Phi^n_\phi \vdash \phi$.
- As an important step of our proof we show that our system can prove that every SCL sentence is equivalent to a sentence in **strong negation normal form**: negation only occurs in front of atomic FO-formulas.

Validity problem for SCL^2

Theorem

For every sentence ϕ *of* SCL^k *there exists a sentence* Ψ *of* ESO^k *such that for every structure* $\mathfrak{A}$ *we have the following equivalence*

$$\text{Verifier does not have a winning strategy in } \mathcal{G}_\infty(\mathfrak{A}, \phi) \Leftrightarrow \mathfrak{A} \models \Psi$$

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Problem

Are the satisfiability problems of BndSCL^2 and SCL^2 decidable? BndSCL^2 has the finite model property and SCL^2 does not have it.

Downward Löwenheim–Skolem Theorem

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Let ϕ be a sentence of BndSCL or SCL and let $\mathfrak{A}$ be a model of ϕ.

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Proof for SCL. Fix a (positional) winning strategy σ for the Verifier in the game $\mathcal{G}_\infty(\mathfrak{A}, \phi)$. We want to construct a countable model $\mathfrak{B}$ so that Eloise has a winning strategy also in the game $\mathcal{G}_\infty(\mathfrak{B}, \phi)$.

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Proof for SCL. Fix a (positional) winning strategy σ for the Verifier in the game $\mathcal{G}_\infty(\mathfrak{A}, \phi)$. We want to construct a countable model $\mathfrak{B}$ so that Eloise has a winning strategy also in the game $\mathcal{G}_\infty(\mathfrak{B}, \phi)$. Pick an arbitrary $b \in A$. We define a sequence of sets $(B_n)_{n\in\mathbb{N}}$ inductively such that, firstly $B_0 = \{b\}$, and then

$$B_{n+1} = B_n \cup \{d \mid \sigma((\exists x\psi, s, +)) = d\},$$

where $\mathrm{range}(s) \subseteq B_n$ and $\exists x\psi \in \mathrm{Subf}(\phi)$. Let $\mathfrak{B}$ be the substructure of $\mathfrak{A}$ induced by the set $\bigcup_{n\in\mathbb{N}} B_n$. $\mathfrak{B}$ is clearly countable. □

Failure of Craig interpolation

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Lemma

For every sentence ϕ of SCL$[\varnothing]$ there exists a finite structure $\mathfrak{A}$ of even size and a finite structure $\mathfrak{B}$ of odd size such that

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Proof.

Follows from the fact that over finite models SCL is contained in $\mathcal{L}^{\omega}_{\omega_1\omega}$. $\square$

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Proof.

Given a binary relation "$<$", there is a SCL sentence ϕ which states that

1. $<$ is a (total) linear order
2. the distance between the smallest and the largest element is finite.

In particular, ϕ projectively defines the class of finite models.

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2. ψ_2 states that E_2 is an equivalence relation with one class of cardinality one while each other class has cardinality two.

Clearly $\phi \wedge \psi_1 \models \neg\psi_2$. However, any interpolant between these sentences needs to distinguish each finite structure of even size from every finite structure of odd size. □

Everywhere determined sentences

Theorem

Let ϕ be a sentence of SCL. If ϕ is determined everywhere, then there exists $n \in \mathbb{N}$ such that ϕ is equivalent with Φ^n_ϕ. In particular, any sentence of SCL which expresses a property that is not FO*-definable is undetermined in some model.*

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If ϕ is determined everywhere, then $\phi \vee \neg\phi$ is valid. This in turn implies that $\Phi^n_{\phi\vee\neg\phi} = \Phi^n_\phi \vee \Phi^n_{\neg\phi}$ is also valid, for some $n \in \mathbb{N}$. We claim that ϕ is equivalent with Φ^n_ϕ. First, we have that $\Phi^n_\phi \models \phi$. Secondly, since $\Phi^n_\phi \vee \Phi^n_{\neg\phi}$ is valid, we have that $\neg\Phi^n_\phi \models \Phi^n_{\neg\phi} \models \neg\phi$. □

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- If we know that ϕ is determined everywhere, then we can effectively find an approximant Φ^n_ϕ which is equivalent with ϕ.

Everywhere determined sentences: the finite case

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2. *ϕ' is determined in every structure.*

Main open problems

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Thanks! :-)