Ordered fragments of first-order logic

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Classical Decision Problem

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The ordered logic OL was introduced independently by Quine and Herzig. The basic idea is that we restrict attention to sentences in which the order of quantification is fixed, and the subformulas need to satisfy an additional uniformity requirement:

$$\forall v_1 \exists v_2 (R(v_1, v_2) \land \forall v_3 (T(v_1, v_2, v_3) \land S(v_1, v_2, v_3)))$$

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Theorem (Herzig, J.)

The satisfiability problem of OL *is* PSPACE*-complete.*

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Other ordered fragment, which has also gained some interest recently, is the fluted logic FL. The underlying idea is that we keep the restriction of OL that variables need to be quantified in a fixed order, but we relax slightly the uniformity requirement:

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Theorem (Pratt-Hartmann, Swast, Tendera)

The satisfiability problem of FL *is* TOWER*-complete.*

Viewing logics as algebras

OL and FL were originally discovered by Quine as a by-product of his attempt to give a variable-free characterization of FO. Quines idea was to introduce a finitely many algebraic operators which would characterize the expressive power of FO.

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Definition (Relational operator)

A k-ary relational operator F is a mapping (proper class) which associates to every set A a function F_A

$$F_A : \mathrm{AD}(A)^k \to \mathrm{AD}(A)$$

Viewing logics as algebras

Definition

Let $\mathcal{F}$ be a set of relational operators and let σ be a relational vocabulary. The set of terms $\mathrm{GRA}(\mathcal{F})[\sigma]$ is defined by the following grammar.

$$\mathcal{T} ::= R \mid F(\mathcal{T}, ..., \mathcal{T}),$$

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Definition

Given a model $\mathfrak{A}$ of vocabulary σ and term $\mathcal{T} \in \mathrm{GRA}(\mathcal{F})[\sigma]$, its interpretation $[\![\mathcal{T}]\!]_{\mathfrak{A}}$ is defined recursively as follows.

1. $[\![R]\!]_{\mathfrak{A}} := R^{\mathfrak{A}}$
2. $[\![F(\mathcal{T}_1, ..., \mathcal{T}_n)]\!]_{\mathfrak{A}} := F_A([\![\mathcal{T}_1]\!]_{\mathfrak{A}}, ..., [\![\mathcal{T}_n]\!]_{\mathfrak{A}})$

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To compare the expressive power of terms and formulas, we note that each first-order formula $\varphi(v_{i_1}, \dots, v_{i_k})$, where $i_1 < \dots < i_k$, defines over each model $\mathfrak{A}$ an AD-relation

$$[\![\varphi]\!]_{\mathfrak{A}} = (\{(a_1, \dots, a_k) \in A^k \mid \mathfrak{A} \models \varphi(a_1, \dots, a_k)\}, k).$$

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Let $\mathcal{F}$ be a set of relational operators and let $X \subseteq \mathrm{FO}$. We say that $\mathrm{GRA}(\mathcal{F})$ and X are equivalent, if for every term $\mathcal{T} \in \mathrm{GRA}(\mathcal{F})$ there exists $\varphi \in X$ so that $[\![\mathcal{T}]\!]_{\mathfrak{A}} = [\![\varphi]\!]_{\mathfrak{A}}$, for every suitable $\mathfrak{A}$, and vice versa.

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Proposition (J.)

$\mathrm{GRA}(\neg, \cap, \exists)$ *and* OL *are sententially equivalent.*

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Proposition (J., Kuusisto)

$\mathrm{GRA}(\neg, \dot{\cap}, \exists)$ *and* FL *are equivalent.*

Extensions of ordered fragments

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To give a concrete example of the type of results we were able to obtain, we define an additional relational operator s as follows. Given an AD-relation (X, k), where $k \geq 2$, we define

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