Undecidability of the halting problem and Gödel's incompleteness theorems

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What is Gödel's incompleteness theorem about?

Theorem (Gödel's first incompleteness theorem, informal version)

If T is a consistent set of axioms whose theorems can be enumerated by a Turing machine, then there are true statements about natural numbers that can not be deduced from T.

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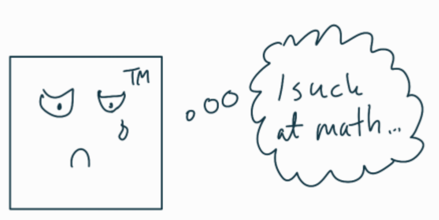

Historical background

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1. 1900: Hilbert's second problem: Prove that arithmetic is consistent by using purely finitistic means.
2. 1930: Gödel proves his incompleteness theorems without having a formal definition of computable function.
3. 1934-1937: Gödel, Church and Turing each introduced models of computation (recursive functions, lambda calculus, Turing machines).

How to prove Gödel's first incompleteness theorem?

D.Hilbert asked if the formal arithmetic (PA: consisting of logic and algebraic axioms and an infinite family of Induction Axioms) can be consistently extended to a complete theory. The question was somewhat vague since an obvious answer was "yes": just add to PA axioms (assumed consistent) a maximal consistent set, clearly existing albeit hard to find. K.Goedel formalized this question as existence among such extensions of recursively enumerable ones and gave it a negative answer. **Its mathematical essence is the absence of total recursive extensions of universal partial recursive predicate.**

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- However, an alternative proof of Gödel's first incompleteness theorem can be given by using the fact that the Halting problem of Turing machines is undecidable.
- **This lecture:** We will formalize one variant of Gödel's first incompleteness theorem and prove it using the aforementioned approach.

First-order logic

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- A standard choice of this formal language is the first-order logic $\mathcal{FO}$. We will start by defining its **syntax** and **semantics**.

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Example

As the vocabulary τ_{Ar} of arithmetic we can choose the set

$$\{0, S, +, \cdot\}$$

1. 0 is a constant.
2. S is a function symbol (the successor function).
3. $+$ and $\cdot$ are function symbols (addition and multiplication).

Syntax of first-order logic

- In addition to a vocabulary, we have an infinite set of variables $\{x, y, z, \dots\}$ and a set of logical symbols

$$\neg, \wedge, \vee, \rightarrow, \exists, \forall$$

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3. $\forall x (x = 0 \vee \exists y (x = S(y)))$

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- We do not formally define the semantics of $\mathcal{FO}$, but it is what you would expect. If a sentence φ of $\mathcal{FO}$ is true in a structure $\mathbb{A}$, then we denote this by $\mathbb{A} \models \varphi$.

Example

1. $\mathbb{N} \models \forall x \forall y (x + y = y + x)$, since addition of natural numbers is commutative.
2. $\mathbb{N} \models \neg \exists x \exists y \exists z (\neg x = 0 \wedge \neg y = 0 \wedge \neg z = 0 \wedge x^3 + y^3 = z^3)$, because Fermat's last theorem is true.
3. $\mathbb{N} \models \forall x (x = 0 \vee \exists y (x = S(y)))$, because every natural number which is not zero has a predecessor.

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- **What are mathematicians doing**: Finding **logical** consequences of T, for various T.
- We say that φ is a **logical consequence** of T, denoted by $T \models \varphi$, if for every structure $\mathbb{A}$ we have that if $\mathbb{A} \models T$, then $\mathbb{A} \models \varphi$.

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Axioms of Peano Arithmetic

$$P_1 \quad \forall x\,(S(x) \neq 0)$$

$$P_2 \quad \forall x \forall y\,(S(x) = S(y) \rightarrow x = y)$$

$$P_3 \quad \forall x\,(x + 0 = x)$$

$$P_4 \quad \forall x \forall y\,(x + S(y) = S(x+y))$$

$$P_5 \quad \forall x\,(x \cdot 0 = 0)$$

$$P_6 \quad \forall x \forall y\,(x \cdot S(y) = (x \cdot y) + x)$$

$$(P_7^{\varphi}) \quad \forall \vec{y}\Big[\big(\varphi(0,\vec{y}) \wedge \forall x(\varphi(x,\vec{y}) \rightarrow \varphi(S(x),\vec{y}))\big) \rightarrow \forall x\,\varphi(x,\vec{y})\Big]$$

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- If rules are effective, which they should be since they are purely formal, then all the proofs of P can be enumerated by a Turing machine!!

Example of an $\mathcal{FO}$-deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\exists x A]^{2} \qquad \dfrac{[A]^{1}}{\exists x A}}{\exists x A \wedge \neg\exists x A}\;\wedge T}{\neg A}\;\neg T, 1}{\forall x \neg A}\;\forall T}{\neg\exists x A \to \forall x \neg A}\;\to T, 2$$

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Complete proof systems

- A remarkable property of $\mathcal{FO}$ is that there exists proof systems for which the following equivalence holds, for every set of $\mathcal{FO}$-sentences T and an $\mathcal{FO}$-sentence φ:

  $$T \models \varphi \iff T \vdash \varphi,$$

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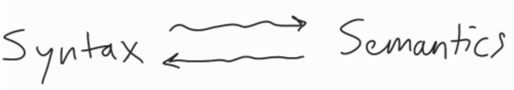

Back to Gödel's incompleteness theorem

Theorem (Gödel's first incompleteness theorem, formal version)

Suppose that T is a computable and consistent set of $\mathcal{FO}$-sentences over the vocabulary τ_{Ar}. Then

$$\{\varphi \in \mathcal{FO} \mid \mathbb{N} \vDash \varphi\} \nsubseteq \{\varphi \in \mathcal{FO} \mid T \vdash \varphi\}$$

Outline of the argument

Let $\mathrm{Th}(\mathbb{N}) := \{\varphi \in \mathcal{FO} \mid \mathbb{N} \vDash \varphi\}$ and $T^{\vdash} := \{\varphi \in \mathcal{FO} \mid T \vdash \varphi\}$.

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2. One can show that there exists a computable mapping which maps each Turing machine M into an arithmetical $\mathcal{FO}$-sentence φ_M so that M halts if and only if $\mathbb{N} \vDash \varphi_M$. (This is the tricky part!)

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    a) We know that we will eventually encounter one of them, since $\mathrm{Th}(\mathbb{N}) \subseteq T^{\vdash}$.
    b) Since T is consistent, we know that it can not contain both φ_M and $\neg\varphi_M$.
4. Thus the halting problem is decidable, which is a contradiction.

Comparison to Gödel's original proof

- In the above argument the use of a diagonalization argument is hidden in the argument that the Halting problem for Turing machines is undecidable.

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- The proof of this is not too difficult, but it becomes more involved if you want to prove that a concrete theory, such as PA, is sufficiently strong. However, this approach naturally yields a proof of Gödel's second incompleteness theorem.

The second incompleteness theorem

Theorem (Gödel's second incompleteness theorem, informal version)

Suppose that $T \supseteq \mathrm{PA}$ is a computable and consistent set of $\mathcal{FO}$-sentences over the vocabulary τ_{Ar}. Then T can not prove that it is consistent.

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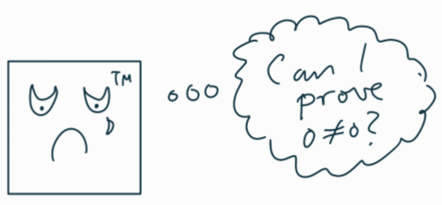
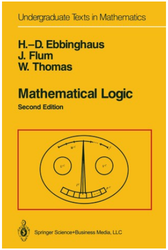


Pointers to literature

- This lecture was partially based on Chapter X in the book



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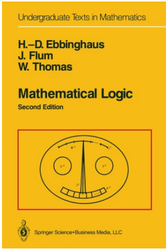

- Panu Raatikainen has an excellent entry on Gödel's incompleteness theorem in Stanford Encyclopedia of Philosophy, which contains several good references:
`https://plato.stanford.edu/entries/goedel-incompleteness/`.

Thanks! :) Questions?

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