Relating Description Complexity to Entropy

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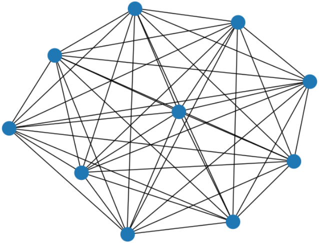
What structures are difficult to describe?



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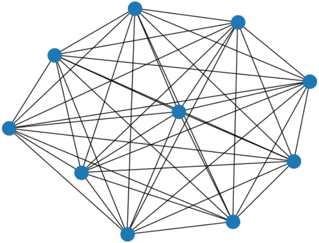

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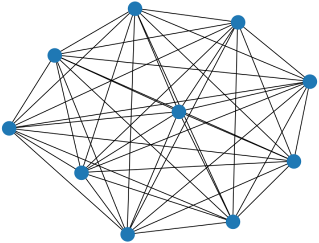

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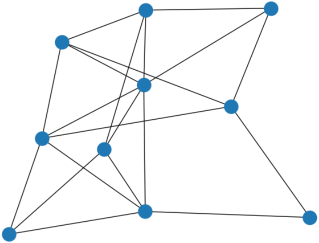

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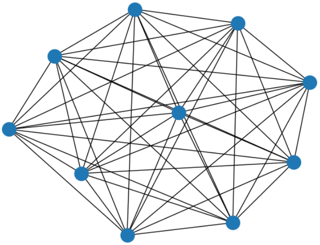

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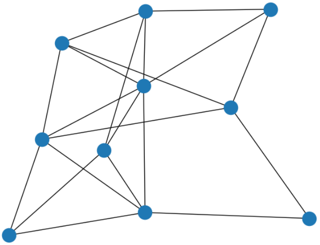

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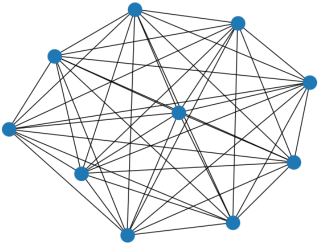

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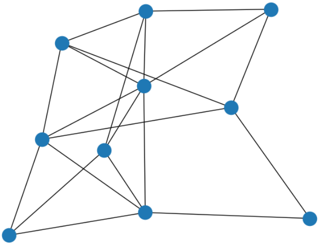

(b) Too difficult...

Intuition: the harder a structure is to describe, the more *random* it is (and vice versa).

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Given $\mathfrak{M} \in \mathcal{M}$ define $[\mathfrak{M}]_{\equiv} := \{\mathfrak{N} \in \mathcal{M} \mid \mathfrak{N} \equiv \mathfrak{M}\}$.

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The $\mathcal{L}$-**description complexity** $C_{\mathcal{L}}(M)$ of $M \in \mathcal{M}/\equiv$ is

$$\min\{\text{size}(\phi) : \mathfrak{N} \vDash \phi \text{ iff } \mathfrak{N} \in M\}$$

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We are especially interested in the setting where $\mathcal{M}$ consists of all τ-models with domain $\{1, \ldots, n\}$, for some fixed relational vocabulary τ.

Most classes have high description complexity

Fix a signature τ containing at least one relation of arity ≥ 2. Let $m = \max\{ar(R) \mid R \in \tau\}$. Set $\mathcal{L} = \mathrm{FO}[\tau]$ and $\mathcal{M}$ = "τ-models with domain $\{1, \ldots, n\}$". In this case $\equiv$ is just the isomorphism relation.

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Proving such a lower bound for an *explicit* isomorphism class seems to be difficult.

Simple fragment of FO

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Definition

Fix a propostional vocabulary $\tau = \{p_1, \ldots, p_k\}$. The set of formulas $\mathrm{GMLU}[\tau]$ is defined by the following grammar:

$$\phi ::= p \mid \neg p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \blacklozenge^{\geq d} \phi \mid \blacksquare^{<d} \phi$$

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Size of a GMLU formula is defined as follows:

1. $\mathrm{size}(p) = \mathrm{size}(\neg p) = 1$.
2. $\mathrm{size}(\psi \vee \chi) = \mathrm{size}(\psi \wedge \chi) = 1 + \mathrm{size}(\psi) + \mathrm{size}(\chi)$.
3. $\mathrm{size}(\blacklozenge^{\geq d} \psi) = \mathrm{size}(\blacksquare^{< d} \psi) = d + \mathrm{size}(\psi)$.

Positive and negative information have the same cost.

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Definition

Given a Kripke model $\mathfrak{M}$ and a world w we define

1. $\mathfrak{M}, w \Vdash \blacklozenge^{\geq d}\psi$ iff there exists $X \subseteq \mathrm{dom}(\mathfrak{M})$ such that $|X| \geq d$ and $\mathfrak{M}, v \Vdash \psi$, for every $v \in X$.
2. $\mathfrak{M}, w \Vdash \blacksquare^{<d}\psi$ iff there exists $X \subseteq \mathrm{dom}(\mathfrak{M})$ such that $|X| < d$ and $\mathfrak{M}, v \Vdash \psi$, for every $v \notin X$.

Lower bounds on description complexity

Given a Kripke model $\mathfrak{M}$ we define

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Theorem (J., Kuusisto & Vilander, 2023)

Let $\mathcal{M}$ be the set of all Kripke models over τ with domain $\{1, \ldots, n\}$.

1. *For every $M \in \mathcal{M}/\equiv$ we have that*

$$C_{\mathrm{GMLU}}(M) \geq \min(n, 2(n-t)),$$

where $t = \max\{t_\pi \mid \pi$ is realized t_π-times in every model in $M\}$.

2. *For every $M \in \mathcal{M}/\equiv$ whose models realize each 1-type sufficiently many times we have that*

$$C_{\mathrm{GMLU}}(M) \geq n + |\tau| 2^{|\tau|+1} - 1.$$

Connection to Boltzmann entropy

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Question

Does a similar result hold in the context of FO?

Connection to expected Boltzmann entropy

$\equiv$ induces a natural probability distribution on $\mathcal{M}$.

$$p_{\equiv}(M) = \frac{|M|}{|\mathcal{M}|}$$

Expected Boltzmann entropy of $\equiv$ is defined as

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Theorem (J., Kuusisto & Vilander, 2023)

Let $\mathcal{M}$ be the set of all Kripke models over τ with domain $\{1, \ldots, n\}$ and $\equiv$ be the isomorphism relation.

1. $H_B(\equiv) \sim |\tau| n$
2. *Letting* $\langle C_{\mathrm{GMLU}} \rangle := \sum_M p_{\equiv}(M) C_{\mathrm{GMLU}}(M)$ *we have that* $\langle C_{\mathrm{GMLU}} \rangle \sim n$.

In particular $H_B(\equiv) \sim |\tau| \langle C_{\mathrm{GMLU}} \rangle$.

Formula size game for GMLU

The formula size game for $\mathrm{GMLU}[\tau]$ has two players: Samson and Delilah. We refer to them as S and D, or he and she, respectively. The game has three parameters: a natural number $r_0 \geq 1$ and two sets of Kripke-models M_0 and N_0. Positions of the game are of the form (r, M, N) and the starting position is (r_0, M_0, N_0).

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