Uniform Guarded Fragments

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Guarded Fragment $\mathcal{GF}$

- **Main idea**: relativize quantification by atoms.

$$\exists x \exists y \exists z (G(x, y, z) \wedge R(x, y) \wedge R(y, z) \wedge R(z, x))$$

Introduced by Andréka, van Benthem and Németi, who where motivated by the standard translation of modal logic into first-order logic $\mathcal{FO}$.

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- $\mathcal{GF}$ shares several desirable properties with modal logic(s): it has a (generalized) tree-model property, its satisfiability problem is decidable, it has the Łoś–Tarski preservation property, ...

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  $$\varphi \models \chi \models \psi$$

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- The two-variable fragment of $\mathcal{GF}$ has CIP [Hoogland & Marx]. **Why?**
- **This talk**: two syntactical restrictions, which we call *uniformity* and *one-dimensionality*, can be used to explain this phenomena.

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- The sentence

$$\exists x \exists y (\exists z (S(x,y,z) \wedge P(z) \wedge x = z) \wedge R(x,y) \wedge S(x,x,y))$$

is uniform, while the sentence

$$\exists x \exists y \exists w (R(x,y) \wedge \exists z S(x,z,w))$$

is not.

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- The sentence

$$\forall x \exists y \exists z (S(x,y,z) \rightarrow (R(x,y) \wedge R(y,z)))$$

is one-dimensional, while the sentence

$$\forall x \forall y (R(x,y) \rightarrow \exists z (S(x,y,z) \wedge R(x,z)))$$

is not.

Uniform one-dimensional fragment of $\mathcal{FO}$

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Theorem (J.)

Uniform one-dimensional $\mathcal{GF}$ ($\mathcal{UGF}_1$) has CIP.

One-dimensional $\mathcal{GF}$ does not have CIP

Consider the sentences

$$\varphi := \exists x \exists y \exists z (G(x,y,z) \wedge R(x,y) \wedge R(y,z) \wedge R(z,x))$$

and

$$\psi := \forall x \forall y (R(x,y) \rightarrow (A(x) \leftrightarrow \neg A(y))).$$

Now $\varphi \vDash \neg\psi$, but there is no interpolant for this entailment.

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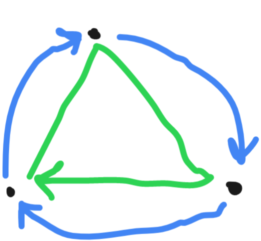

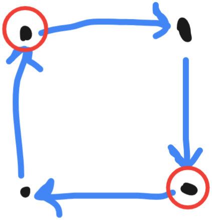

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Consider the sentences

$$\varphi := \exists x \exists y (T(x,y) \wedge \exists z R(x,y,z) \wedge \exists z S(x,y,z))$$

and

$$\psi := \forall x \forall y \forall z (R(x,y,z) \rightarrow (P(y) \leftrightarrow Q(x)))$$
$$\wedge \forall x \forall y \forall z (S(x,y,z) \rightarrow (P(y) \leftrightarrow \neg Q(x))).$$

Now $\varphi \vDash \neg\psi$, but there is no interpolant for this entailment.

Uniform $\mathcal{GF}$ does not have CIP

Consider the sentences

$$\varphi := \exists x \exists y (T(x,y) \land \exists z R(x,y,z) \land \exists z S(x,y,z))$$

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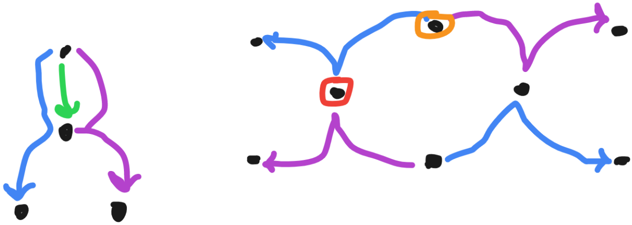
$$\psi := \forall x \forall y \forall z (R(x,y,z) \to (P(y) \leftrightarrow Q(x)))$$
$$\land \forall x \forall y \forall z (S(x,y,z) \to (P(y) \leftrightarrow \neg Q(x))).$$

Now $\varphi \models \neg\psi$, but there is no interpolant for this entailment.



How to prove that $\mathcal{UGF}_1$ has CIP

Lemma

Suppose that $\varphi, \psi \in \mathcal{UGF}_1$. *Suppose that there is no* $\chi \in \mathcal{UGF}_1$ *such that* $\varphi \models \chi \models \psi$ *and* $\mathrm{sig}(\chi) \subseteq \mathrm{sig}(\varphi) \cap \mathrm{sig}(\psi)$. *Then there are models* $\mathcal{A}$ *and* $\mathcal{B}$ *such that* $\mathcal{A} \models \varphi, \mathcal{B} \models \psi$ *and*

$$\mathcal{A} \sim^{\mathcal{UGF}_1}_{\mathrm{sig}(\varphi)\cap\mathrm{sig}(\psi)} \mathcal{B}.$$

How to prove that $\mathcal{UGF}_1$ has CIP

Lemma

Suppose that $\varphi, \psi \in \mathcal{UGF}_1$. Suppose that there is no $\chi \in \mathcal{UGF}_1$ such that $\varphi \vDash \chi \vDash \psi$ and $\mathrm{sig}(\chi) \subseteq \mathrm{sig}(\varphi) \cap \mathrm{sig}(\psi)$. Then there are models $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \vDash \varphi, \mathcal{B} \vDash \psi$ and

$$\mathcal{A} \sim^{\mathcal{UGF}_1}_{\mathrm{sig}(\varphi)\cap\mathrm{sig}(\psi)} \mathcal{B}.$$

Given such structures $\mathcal{A}$ and $\mathcal{B}$, we will construct a third structure $\mathcal{U}$ (the amalgam) such that

$$\mathcal{A} \sim^{\mathcal{UGF}_1}_{\mathrm{sig}(\varphi)} \mathcal{U}$$

and

$$\mathcal{B} \sim^{\mathcal{UGF}_1}_{\mathrm{sig}(\psi)} \mathcal{U},$$

which shows that $\varphi \wedge \neg\psi$ is satisfiable.

Conclusions and future directions

- One-dimensionality and uniformity can be used to explain – at least partially – why $\mathcal{GF}$ does not have CIP while several other modal logics do have it.
- Further extensions of $\mathcal{UGF}_1$ should be studied, for example with counting quantifiers.
- More constructive proofs for the existence of interpolant are needed (Tableu-algorithms etc.).

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Thanks! :-)