What is a fragment?

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Introduction

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- The research around this question has remained active to this day, but is currently **in my opinion** stuck: the field has a large body of results but it is lacking direction.
- An almost obvious research direction candidate would be to classify fragments of first-order logic based on whether they are decidable or not. However, to make a formal conjecture, we need to first specify what fragments we want to study.

This talk

1. Short introduction to the satisfiability problem
2. Examples of fragments
3. Classification of fragments
4. Expressive power vs. decidability

The satisfiability problem

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- A lot of research around *fragments* of $\mathcal{FO}$, i.e. subsets of $\mathcal{FO}$. The goal is to find expressive fragments with a decidable satisfiability problem.

Examples of fragments

- Prefix fragments: fix a family $\mathcal{F}$ of prefixes and consider the set of sentences of the form

$$Q_1 x_1 \dots Q_n x_n \psi(x_1, \dots, x_n),$$

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- Several other fragments known: monadic first-order logic, guarded fragment, triguarded fragment, guarded negation fragment, uniform one-dimensional fragment, fluted logic, ordered logic, Maslov fragment, Herbrand fragment, positive first-order logic, ...

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- But on the other hand we are not interested in every single fragment. What fragments are interesting to us?
  - Difficult to answer since any answer should cover fragments with quite distinct syntax (e.g. prefix-fragments versus two-variable logic).
- We can try to come up with natural requirements that a fragment should satisfy.

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- It turns out that if we require that a reasonable fragment has a recursive syntax, then the above claim holds, because the set of valid sentences of $\mathcal{FO}$ is recursively enumerable.
- However, it does not hold if we restrict our attention to finite models:

$$\{\bot\} \cup \{\varphi^n \mid \varphi \in \mathcal{FO} \text{ has a model of size } n\}$$

Here

$$\varphi^n := \underbrace{\varphi \wedge \cdots \wedge \varphi}_{n\text{-times}}$$

Decidability and expressive power

- To exclude the previous counterexample, we will require that a reasonable fragment $\mathcal{L}$ should be effectively closed under conjunction: there should exists a computable function $f : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ so that $f(\varphi, \psi)$ is equivalent with $\varphi \wedge \psi$.

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- One can show that there exists no recursive subset $\mathcal{L}$ of $\mathcal{FO}$ which has the same expressive power as $\mathcal{FO}$ over finite models, is effectively closed under conjunction and has a decidable finite satisfiability problem.

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